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An Application of dispersive partial differential equations in mathematics

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Abstract

In this work, fractional variational iteration method (FVIM) has been applied effectively to track down the arrangement of fragmentary dispersive fractional differential conditions of third-request in complex spaces. The thought about charts make sense of the personality of the answer for various values of fragmentary request. The strength and accurateness of the proposed procedures are examined with the assistance of two test models.

Keywords: *Fractional variational iteration method, dispersive partial differential equations, Caputo fractional derivative,*

Introduction

Definition 2.1. *A real function* $f(\mathrm{t}), \mathrm{t}>0$ *is said to be in the space* $C_\alpha, \alpha \in R$ *if there exists a real number* $p(>\alpha)$ *, such that* $f(\mathrm{t})=\mathrm{t}^p f_1(\mathrm{t})$ *where* $f_1 \in C[0,\infty]$*. clearly* $C_\alpha \subset C_\beta$ *if* $\beta \le \alpha$ *[48].*

Definition 2.2. *A function* $f(\mathrm{t}), \mathrm{t}>0$ *is said to be in the space* $C_\alpha^m, m \in N \cup \{0\}$*, if* $f^{(\mathrm{m})} \in C_\alpha$ *.*

Definition 2.3. *The left sided Riemann-Liouville fractional integral of order* $\mu>0$ *, [48-51] of a function* $f \in C_\alpha, \alpha \ge -1$ *is defined as:*

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\mu}} d\tau = \frac{1}{\Gamma(\mu+1)} \int_0^t f(\tau)(d\tau)^\mu$$

$$I^0 f(t) = f(t).$$

Definition 2.4. The (left sided) Caputo fractional derivative of $f, f \in C_{-1}^m, m \in \mathrm{IN} \cup \{0\}$ [48-51],

$$D_t^\mu f(t) = \begin{cases} [I^{m-\mu} f^{(m)}(t)], & m-1<\mu<m, m \in \mathrm{IN}, \\ \frac{d^m}{\mathrm{d}t^m} f(t), & \mu = m. \end{cases}$$

a. $I_t^\alpha f(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(x,s)\mathrm{d}s, \alpha, t>0.$

b. $D_t^\alpha u(x,t) = I_t^{m-\alpha} \frac{\partial^m u(x,t)}{\partial t^m} f(t), m-1<\alpha<m.$

c. $I^\mu t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\mu+\gamma+1)} t^{\mu+\gamma}.$

Definition 2.5. *The fractional trigonometric function [52] is denoted by*

$$E_\alpha(i^\alpha t^\alpha) = cos_\alpha t^\alpha + i^\alpha sin_\alpha t^\alpha,$$

where

$$cos_\alpha t^\alpha = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n\alpha}}{\Gamma(1+2n\alpha)}$$

and $sin_\alpha t^\alpha = \sum_{n=0}^{\infty}(-1)^n \frac{t^{(2n+1)\alpha}}{\Gamma(1+(2n+1)\alpha)}$.

In particular when $\alpha = 1$ *above equations reduces to*

$cos\, t = \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{\Gamma(1+2n)}$ *and*

$sin\, t = \sum_{n=0}^{\infty}(-1)^n \frac{t^{(2n+1)}}{\Gamma(2n+2)}$ *respectively.*

3 The Proposed FVIM method for the Fractional third-order dispersive partial differential equation

To define solution process of third-order fractional dispersive partial differential equation by using fractional variational iteration method, we study the ensuing fractional differential equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + 2\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial y^3} = 0, 0 < \alpha \leq 1. \quad (3.1)$$

According to the FVIM, a correction functional ([3]) can be built for above equation as

$$u_{n+1}(\mathrm{x,t}) = u_n + \frac{1}{\Gamma(1+\alpha)}\int_0^t \lambda\left(\frac{\partial^\alpha u_n}{\partial \tau^\alpha} + 2\frac{\partial^3 \tilde{u}_n}{\partial x^3} + \frac{\partial^3 \tilde{u}_n}{\partial y^3}\right)(\mathrm{d\tau})^\alpha. \quad (3.2)$$

Now by the variational theory λ must satisfy

$\frac{\partial^\alpha \lambda}{\partial \tau^\alpha} = 0$ and $1 + \lambda|_{\tau=t} = 0$. From these equations, we obtain $\lambda = -1$ and a new correction functional

$$u_{n+1}(x,t) = u_n - \frac{1}{\Gamma(1+\alpha)}\int_0^t \left(\frac{\partial^\alpha u_n}{\partial \tau^\alpha} + 2\frac{\partial^3 u_n}{\partial x^3} + \frac{\partial^3 u_n}{\partial y^3}\right)(\mathrm{d\tau})^\alpha. \quad (3.3)$$

We can build consecutive iterations $u_n, n \geq 0$ after by using λ, a common Lagrange's multiplier, that can be obtained by variational theory. The functions $\tilde{u}_n$ is restricted variation that means $\delta\tilde{u}_n = 0$. Consequently, first we elect the Lagrange multiplier λ, which can be obtained using integration by parts. In this way we can obtain sequences $u_{n+1}(x,t), n \geq 0$ of the solution and finally the exact solution can be obtained as $u(x,t) = \lim_{n\to\infty} u_n(x,t)$.

Numerical Experiments

Example 4.1. *We study the ensuing time-fractional dispersive partial differential equation:*
$u_t^{\alpha} + 2u_x + u_{xxx} = 0, t > 0, 0 < \alpha \leq 1,$

subject to the initial condition
$u(x, 0) = \sin x.$

In particular when $\alpha = 1$ *the exact solutions of Eq. (4.4) is* $u(x, t) = \sin(x - t)$.
By given initial condition, we can take initial solutions as

$$u_0 = \sin(x),$$
$$u_1(x,t) = u_0 - \frac{1}{\Gamma(1+\alpha)}\int_0^t \left(\frac{\partial^{\alpha} u_0}{\partial \tau^{\alpha}} + 2\frac{\partial u_0}{\partial x} + \frac{\partial^3 u_0}{\partial x^3}\right)(d\tau)^{\alpha} = \sin(x) - \frac{t^{\alpha}\cos(x)}{\Gamma(\alpha+1)},$$
$$u_2(x,t) = \sin(x) - \frac{t^{\alpha}\cos(x)}{\Gamma(\alpha+1)} - \frac{t^{2\alpha}\sin(x)}{\Gamma(2\alpha+1)},$$
$$u_3(x,t) = \sin(x) - \frac{t^{\alpha}\cos(x)}{\Gamma(\alpha+1)} - \frac{t^{2\alpha}\sin(x)}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}\cos(x)}{\Gamma(3\alpha+1)},$$
$$u_4(x,t) = \sin(x) - \frac{t^{\alpha}\cos(x)}{\Gamma(\alpha+1)} - \frac{t^{2\alpha}\sin(x)}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}\cos(x)}{\Gamma(3\alpha+1)} + \frac{t^{4\alpha}\cos(x)}{\Gamma(3\alpha+1)},$$
$$u_n(x,t) = \sin x\left(1 - \frac{t^{2\alpha}}{\Gamma(\alpha+1)} + \ldots \frac{(-1)^n t^{2\alpha n}}{\Gamma(2n\alpha+1)}\right) - \cos x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \cdots + \frac{(-1)^{n+1}(t^{\alpha})^{2n-1}}{\Gamma((2n-1)\alpha+1)}\right),$$

Table 1: Absolute error for different values of fractional order α.

		$\alpha = 1$	$\alpha = 0.75$	$\alpha = 0.5$	$\alpha = 0.25$
x	t	$\lvert u - u_{10}\rvert$	$\lvert u - u_{10}\rvert$	$\lvert u - u_{10}\rvert$	$\lvert u - u_{10}\rvert$
-10	0.2	00	00	4.44×10^{-16}	3.8×10^{-7}
-10	0.4	2.22×10^{-16}	5.55×10^{-17}	7.11×10^{-13}	1.74×10^{-5}
-10	0.6	1.11×10^{-16}	00	6.35×10^{-11}	1.62×10^{-4}
-10	0.8	2.22×10^{-16}	2.22×10^{-16}	1.54×10^{-9}	7.86×10^{-4}
-10	1	2.22×10^{-16}	7.33×10^{-15}	1.82×10^{-8}	2.67×10^{-3}
00	0.2	00	00	1.11×10^{-16}	1.8×10^{-7}
00	0.4	00	00	1.88×10^{-13}	9.1×10^{-6}
00	0.6	00	1.11×10^{-16}	1.96×10^{-11}	8.96×10^{-5}
00	0.8	00	1.11×10^{-16}	5.28×10^{-10}	4.52×10^{-4}
00	1	00	1.22×10^{-15}	6.76×10^{-9}	1.58×10^{-3}
10	0.2	1.11×10^{-16}	00	2.22×10^{-16}	7.85×10^{-8}
10	0.4	00	5.55×10^{-17}	3.96×10^{-13}	2.12×10^{-6}
10	0.6	1.11×10^{-16}	00	3.06×10^{-11}	1.16×10^{-5}
10	0.8	2.22×10^{-16}	5.55×10^{-17}	6.55×10^{-10}	2.77×10^{-5}
10	1	5.55×10^{-17}	5.22×10^{-15}	6.9×10^{-9}	1.91×10^{-5}

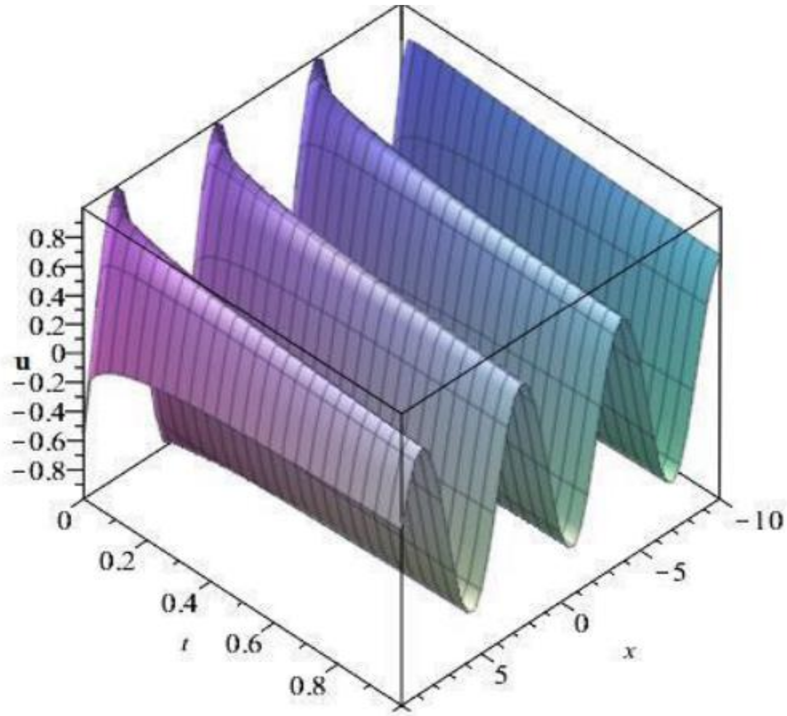

Figure 1(a): Surface shows approximate solution for $\alpha = 0.25$.

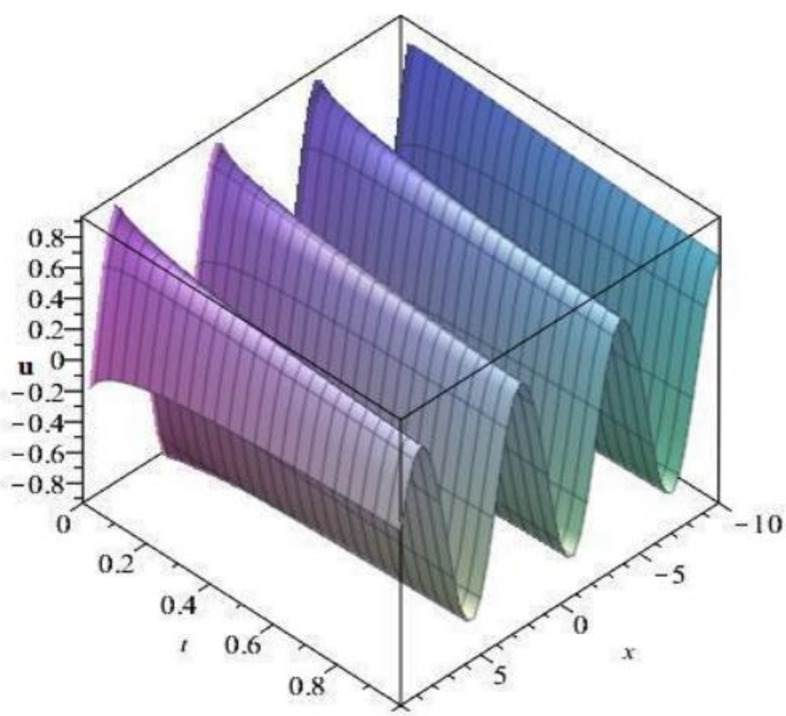

Figure 1(b): Surface shows exact solution for $\alpha = 0.25$.

Conclusion

In this paper, Fractional variation iteration method (FVIM) has been applied effectively for addressing time-partial third-request dispersive halfway differential conditions. It is clearly seen that FVIM is an extremely proficient and strong mathematical strategy to get the estimated arrangement. The technique is utilized in an immediate manner without utilizing adomain polynomial, linearization, irritation or prohibitive presumptions. Thusly, FVIM is simpler and more advantageous than different strategies.

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